2019-Sep-17 (lecture)

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MAT 844 - 2019

Outline:

- Exact differentials
- Integrating Factors
- Checking solutions
- Bernoulli Equation
- Qualitative Methods

Last time:

We defined an exact differential $P(x,y)\,dx + Q(x,y)\,dy = 0$ if $\exists f(x,y)$ s.t. $P(x,y) = \frac{\partial}{\partial x} f(x,y)$, $Q(x,y) = \frac{\partial}{\partial y} f(x,y)$.

We can recognize it by $\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial x} Q(x,y)$

Recognizing is easy. But how can we solve for $f(x,y)$?

Yes. (Tenenbaum 9.43) Given $P(x,y)\,dx + Q(x,y)\,dy = 0$ an exact differential,

$$f(x,y) = \int_{x_0}^{x} P(\bar{x}, y)\,d\bar{x} + \int_{y_0}^{y} Q(x_0, \bar{y})\,d\bar{y}, \text{ where}$$

the line segments $(x_0, y_0) - (x, y_0)$ and $(x_0, y_0) - (x_0, y)$ lie entirely in R, R is simply connected.

Ex.

$$\left(\frac{2xy+1}{y}\right)dx + \left(\frac{y-x}{y^2}\right)dy = 0$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{2xy+1}{y}\right) &= \frac{\partial}{\partial y}\left(2x + \frac{1}{y}\right) = -\frac{1}{y^2} \\ \frac{\partial}{\partial x}\left(\frac{y-x}{y^2}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{y^2}\right) = -\frac{1}{y^2} \end{aligned}\right\} \text{exact}$$

So long as $y > 0$, all the partials exist and are continuous.

We choose $x_0 = 0$, $y_0 = 1$

$$f(x,y) = \int_0^x \frac{2\bar{x}y+1}{y}\,d\bar{x} + \int_1^y \frac{\bar{y}-0}{\bar{y}^2}\,d\bar{y}$$

$$= \int_0^x 2\bar{x}\,d\bar{x} + \int_0^x \frac{1}{y}\,d\bar{x} + \int_1^y \frac{1}{\bar{y}}\,d\bar{y}$$

$$= \int_0^x 2\bar{x}\,d\bar{x} + \int_0^x \frac{1}{y}\,d\bar{x} + \int_1^y \frac{1}{\bar{y}}\,d\bar{y}$$

$$= x^2 + \frac{x}{y} + \ln y$$

Then $f(x,y) = x^2 + \frac{x}{y} + \ln y = C$ solves the ODE for $y > 0$.

$$2x\,dx + \frac{1}{y}dx + \frac{x}{-y^2}dy + \frac{1}{y}dy = 0$$

$\boxed{d(xy) = y\,dx + x\,dy}$ $\longrightarrow$

$$\left(2x + \frac{1}{y}\right)dx + \left(\frac{1}{y} - \frac{x}{y^2}\right)dy = 0 \quad \checkmark$$

Alternately, can try solving for $f(x,y)$ (heuristic)

$$(3e^{3x}y - 2x)\,dx + e^{3x}\,dy = 0$$

Is this exact?

$$\frac{\partial}{\partial y}(3e^{3x}y - 2x) = 3e^{3x}$$

$$\frac{\partial}{\partial x} e^{3x} = 3e^{3x}$$

" ✓ Yes, exact

$$f(x,y) = \int (3e^{3x}y - 2x)\,dx = e^{3x}y - x^2 + C_1(y)$$

$$f(x,y) = \int e^{3x}\,dy = e^{3x}y + C_2(x)$$

$$\underline{e^{3x}}y - x^2 + C_1(y) = \underline{e^{3x}}y + C_2(x)$$

$$-x^2 + C_1(y) = C_2(x)$$

$$\Rightarrow C_1(y) = 0, \quad C_2(x) = -x^2$$

$$\Rightarrow f(x,y) = e^{3x}y - x^2$$

$\leftarrow$ double check soln exercise for reader

So, $e^{3x}y - x^2 = C$

Also, can sometimes look up common integrable combinations in Tenenbaum, Table, Lesson 10A.

Integrating Factors

Sometimes, given an inexact $P(x,y)\,dx + Q(x,y)\,dy = 0$, Sometimes, given an inexact $P(x,y)dx + Q(x,y)dy = 0$, we can convert it into an exact ODE by multiplying.

Def. A integrating factor IF will convert an inexact ODE $P(x,y)dx + Q(x,y)dy = 0$ into an exact ODE $\text{IF}\, P(x,y)dx + \text{IF}\, Q(x,y)\, dy = 0$

Ex. $(y^2+y)dx - x\, dy = 0$

$$\frac{\partial}{\partial y}(y^2+y) = 2y+1 \neq \frac{\partial}{\partial x}(-x) = -1$$

not exact

However, if we multiply by y^{-2} ← IF

$$y^{-2}(y^2+y)\, dx - y^{-2} x\, dy = 0$$

$$\Rightarrow (1+y^{-1})\, dx - xy^{-2} dy = 0$$

$$\frac{\partial}{\partial y}(1+y^{-1}) = -\frac{1}{y^2} = \frac{\partial}{\partial x}(-xy^{-2}) = -\frac{1}{y^2}$$

exact

Mentimeter: find an IF for

Ex. $xy\, dx + (1+x^2)dy = 0$ $\quad$ IF $= y$

$$xy^2 dx + (1+x^2)y\, dy = 0$$

$$\frac{\partial}{\partial y} xy^2 = 2xy \qquad \frac{\partial}{\partial x}(1+x^2)y = 2xy$$

Ex. $(x^2+y^2+x)\, dx + xy\, dy = 0$ $\quad$ IF $= x$

Integrating Factors are generally hard to find, so we will not spend that much time on them.

However, some common types of ODEs have known IFs.

Linear ODE first order Given $\frac{dy}{dx} + P(x)y = Q(x)$,

a known IF is $e^{\int P(x)dx}$

Linear ODE first order (Lesson 11 B)

Given $\frac{dy}{dx} + P(x)y = Q(x)$

a known IF is $e^{\int P(x)dx}$.

Ex. $y' + y\cos x = \frac{1}{2}\sin 2x$

$$\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$$

Let $IF = e^{\int \cos x\, dx} = e^{\sin x}$.

$$\frac{dy}{dx} e^{\sin x} + y\cos x\, e^{\sin x} = \frac{1}{2}\sin 2x\, e^{\sin x}$$

$$dy\, e^{\sin x} + dx\left(y\cos x\, e^{\sin x} - \frac{1}{2}\sin 2x\, e^{\sin x}\right) = 0$$

$$dx\, e^{\sin x}\left(y\cos x - \frac{1}{2}\sin 2x\right) + dy\, e^{\sin x} = 0$$

Is this exact?

$$\frac{\partial}{\partial y}\left(y\cos x - \frac{1}{2}\sin 2x\right)e^{\sin x} = \cos x\, e^{\sin x}$$

$$\frac{\partial}{\partial x}\left(e^{\sin x}\right) = \cos x\, e^{\sin x}$$

$=$ → exact.

What is $f(x,y)$

Recall $f(x,y) = \int_{x_0}^{x} P(\bar{x}, y)\, d\bar{x} + \int_{y_0}^{y} Q(x_0, \bar{y})\, d\bar{y}$

or

→ $f(x,y) = \int_{x_0}^{x} P(\bar{x}, y_0)\, d\bar{x} + \int_{y_0}^{y} Q(x, \bar{y})\, d\bar{y}$

Let $x_0 = 0$, $y_0 = 0$

$$f(x,y) = \int_0^x \left(0\cos\bar{x} - \frac{1}{2}\sin 2\bar{x}\right)e^{\sin\bar{x}}\, dx + \int_0^y e^{\sin x}\, d\bar{y}$$

$$= \int_0^x \left(-\frac{1}{2}\sin 2\bar{x}\right)e^{\sin\bar{x}}\, dx + y e^{\sin x}$$

$$= -\int_0^x \sin\bar{x}\cos\bar{x}\, e^{\sin\bar{x}}\, d\bar{x} + y e^{\sin x}$$

Let $u = \sin\bar{x}$ $\qquad = -\int_0^{\sin x} u e^u\, du + y e^{\sin x}$

$du = \cos\bar{x}\, d\bar{x}$

Let $u = \sin\hat{x}$ $\quad = -\int_0^{\sin x} ue^u\,du + ye^{\sin x}$

$du = \cos\hat{x}\,d\hat{x}$

$$= -e^u(u-1)\Big|_0^{\sin x} + ye^{\sin x}$$

$$f(x,y) = -e^{\sin x}(\sin x - 1) - 1 + ye^{\sin x} = C$$

$$ye^{\sin x} = C + e^{\sin x}(\sin x - 1) + 1$$

$$y = Ce^{-\sin x} + (\sin x - 1)$$

$\boxed{C+1 \to C}$

Checking solutions: $\quad y' + y\cos x = \frac{1}{2}\sin 2x$

Is $y = 5e^{-\sin x} + \sin x - 1$ a solution? $\quad (C=5)$

$$y' = -5\cos x\, e^{-\sin x} + \cos x$$

$$-5\cos x\, e^{-\sin x} + \cos x + \left(5e^{-\sin x} + \sin x - 1\right)\cos x = \frac{1}{2}\sin 2x$$

$$\sin x\cos x = \frac{1}{2}\sin 2x \quad ✓$$

Going backwards from solution:

Ex. $\quad y = Ce^{-\sin x} + \sin x - 1$, a 1-parameter family of solutions

Differentiating, $y' = -C\cos x\, e^{-\sin x} + \cos x$ is clearly a solution, but it contains an arbitrary constant.

Let's get rid of the constant

$$y\cos x + y' = C\cos e^{-\sin x} + \sin x\cos x - \cos x$$
$$- C\cos e^{-\sin x} + \cos x$$

$$y\cos x + y' = \sin x\cos x = \frac{1}{2}\sin 2x$$

Ex. Sometimes we can just use implicit differentiation to directly get rid of the constant.

Given sol. $x^2 + y^2 = C$, a 1-parameter family of solutions

$\Rightarrow 2x + 2yy' = 0$

Given sol. $x^2+y^2=C$, a 1-parameter family of solutions,

$$\Rightarrow 2\cancel{x} + 2yy' = 0$$

$$x + yy' = 0$$ ← ODE with $x^2+y^2=C$ as family of solutions.

Or $x^2+y^2=C$

$$d(x^2+y^2)=0$$

$$2x\,dx + 2y\,dy = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2x + 2yy' = 0$$

Bernoulli Equation

A special type of ODE given by $\frac{dy}{dx} + P(x)y = Q(x)y^n$

To solve it, multiply by $(1-n)y^{-n}$

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx} + (1-n)y^{1-n}P(x) = (1-n)Q(x)$$

Substitute $u = y^{1-n}$, $du = (1-n)y^{-n}dy$

$$\Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

This is now a first-order ODE, so we can use IF $e^{\int(1-n)P(x)\,dx}$.

Ex. $y' + xy = \frac{x}{y^3}$, $y \neq 0$

multiply by $4y^3 \Rightarrow 4y^3y' + 4xy^4 = 4x$

substitute $u = y^4$, $du = 4y^3dy \Rightarrow u' + 4ux = 4x$

IF $= e^{\int 4x\,dx} = e^{2x^2} \Rightarrow e^{2x^2}u' + 4e^{2x^2}(ux - x) = 0$

exact differential $\Rightarrow e^{2x^2}du + 4e^{2x^2}(ux-x)\,dx = 0$

$$\Rightarrow ue^{2x^2} - e^{2x^2} = C$$

$$ue^{2x^2} = C + e^{2x^2}$$

$$u = Ce^{-2x^2} + 1$$

$$ue^{2x^2} = C + e^{2x^2}$$
$$u = Ce^{-2x^2} + 1$$
$$y^4 = Ce^{-2x^2} + 1$$

(Tesch 1.15
Notice, of Bernoulli type
and also separable)

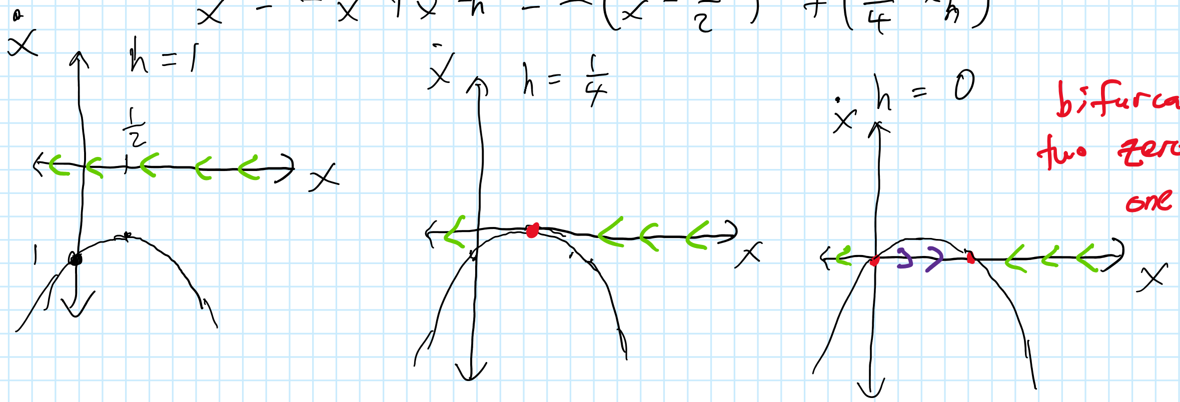
Qualitative methods

Autonomous

Consider the logistic growth model

$$\dot{x} = (1-x)x - h$$

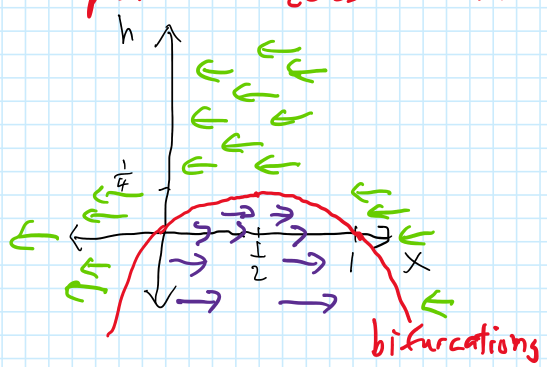
$$\dot{x} = -x^2 + x - h = -\left(x - \tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4} - h\right)$$



bifurcation,
two zeros, one stable,
one unstable.

Let's plot the zeros of x as a function of h

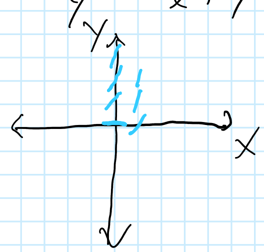
$$(1-x)x - h = 0$$
$$h = (1-x)x$$



Note, some texts like to flip the axis.

Direction Fields

$y' = x + y$ (or $\dot{x}(t) = x + t$)

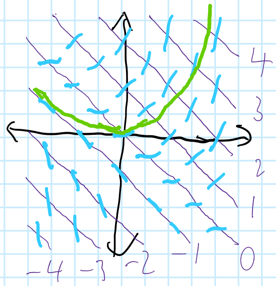


We can plot the slope at every point.

Slow and tedious

$\checkmark$

Def. An isocline as a curve where $y' = c$, for c a constant.



$c = x + y$

$y = -x + c$

Integral curves can be plotted tangent to each of the slope lines

$y = e^x - x - 1$

Can also plot in Matlab, WolframAlpha, etc.